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Trapped modes in particles with a negative refractive index

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The natural oscillations of the electromagnetic field in a particle made from left-handed metamaterial, where both permittivity and permeability are negative, are considered. Based on the exact solution of the sourceless Maxwell equations, it is shown that due to the opposite directions of the phase and group velocities in the metamaterial, natural oscillations in such particles decay exponentially at infinity, that is, these natural oscillations can be considered as trapped modes with a finite energy. The manifestation of such modes in experiments with Bessel beams is also discussed.

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At present, the optical community pays great attention to systems where the existence of electromagnetic fields strongly localized in space is possible at frequencies allowing the waves to propagate in a free space. Such states are often called bound states in the continuum (BIC) [1,2]. In two-dimensional systems and photonic crystals, a number of such states have been found [3–5]. The situation with localized states of light in three-dimensional particles is much more complicated. The point is that the usual quasi-normal modes for three-dimensional particles of a limited volume grow exponentially at infinity [6] and therefore, in principle, are not square integrable.

The decaying at infinity solutions of the homogeneous Maxwell equations in the presence of particles of limited volume exists, but they are little known since they are based on non-standard definitions of eigenmodes [7–11]. First of all, there are ε modes, which are the solutions of the sourceless Maxwell equations, where the eigenvalue is the permittivity of the particle [7–9] rather than the frequency (as in ordinary quasi-normal modes). For finding ε modes, as well as for finding quasi-normal modes, the Sommerfeld radiation condition [6] is used. Decay of the ε mode at infinity is related to the complex values of the eigenvalues of permittivity, corresponding to the active medium inside the resonator, compensating the radiation losses [9]. It is very important that ε modes form a complete orthogonal system of functions and, therefore, are extremely useful for describing the properties of particles of a given shape made from an arbitrary material.

Another type of modes decaying at the infinite distance from three-dimensional particles is represented by recently discovered perfectly non-radiating or invisibility modes [9–11]. These modes are solutions of Maxwell's equations in the presence of a dielectric particle obtained beyond the restriction imposed by the Sommerfeld radiation condition. These modes exist for real eigenfrequencies and therefore fundamentally have no radiation losses.

Both ε modes and perfect non-radiating modes have several attractive properties, but the decay of fields at spatial infinity in these modes (as 1/r) is not fast enough and does not provide their square integrability.

The possibility of creating spherical resonators with highly localized modes has been also studied in Refs. [12,13], where it is proposed to use shells made of epsilon-near-zero (ENZ) materials to suppress radiation. The resonators considered in these works should, apparently, be attributed not to open but to closed resonators, and the boundary conditions arising due to ENZ layers should be considered as a generalization of the boundary conditions of a perfect electrical conductor (PEC) or a perfect magnetic conductor (PMC).

The only non-trivial example of localized states in a three-dimensional space that have a finite energy (known to the authors) is the Neumann–Wigner strange modes [14] and their generalizations [15,16]. These strange modes have a finite energy despite the fact that this energy is above the barrier. However, the price for a strong localization of the field is a non-trivial potential that, oscillating, extends to infinity.

Thus, as far as the authors know, strongly localized 3D finite-energy modes with a potential localized in space are still unknown for open resonators. In this Letter, we will show that highly localized modes in 3D particles of limited volume are possible if these particles are made of the metamaterial where both ε and μ are negative (left-handed or double-negative (DNG) metamaterial) [17,18]. From a practical point of view, it is very important that the incorporation of gain material in the high-local-field areas of a metamaterial makes it possible to fabricate an extremely low-loss and active optical DNG metamaterials [19]. Such materials may theoretically exist as natural media up to the THz frequency region [20].

Particles made of the metamaterials with a negative refractive index have anomalous optical properties [21–23], and, therefore, it is natural to expect that natural oscillations of light in such particles will also have unusual properties.

In this Letter, by the example of spherical DNG particles, we show that in any particles with a negative refractive index, there are modes with a complex frequency with the fields decaying exponentially at spatial infinity, and, therefore, it is natural to call these modes trapped modes.

The sourceless Maxwell equations in the presence of DNG particles have the usual form [24]:

$$\begin{aligned} &\nabla\times\mathbf{E}_1 = ik_0\mu\mathbf{H}_1; \nabla\times\mathbf{H}_1 = -ik_0\varepsilon\mathbf{E}_1, \text{inside particle},\\ &\nabla\times\mathbf{E}_2 = ik_0\mathbf{H}_2; \nabla\times\mathbf{H}_2 = -ik_0\mathbf{E}_2, \text{ outside particle},\\ &\mathbf{n}\times(\mathbf{E}_1-\mathbf{E}_2)=0; \mathbf{n}\times(\mathbf{H}_1-\mathbf{H}_2)=0, \quad \text{at the boundary}, \end{aligned} \tag{1}$$

where ε and μ are the particle relative permittivity and permeability, $k_0 = \omega/c$ is the wavenumber in vacuum, ω is the frequency, c is the speed of light, $\mathbf{n}$ is the unit normal vector to the boundary, and $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{H}_1$, and $\mathbf{H}_2$ are corresponding field strengths. Throughout the Letter, it is assumed that the dependence of fields on time has the form $\exp(-i\omega t)$.

Without any additional conditions, the solutions of Maxwell's equations [Eq. (1)] form a continuum of solutions with very different behavior at infinity. To find localized eigenoscillations, one usually imposes the Sommerfeld radiation conditions at infinity [6]:

$$\mathbf{E}_2(\mathbf{r}, k_0) \to \frac{\exp(ik_0 r)}{r}\mathbf{F}(\mathbf{k}), \tag{2}$$

where $\mathbf{r}$ is the position vector, $r = |\mathbf{r}|$, $\mathbf{F}(\mathbf{k})$ is the scattering amplitude, and $\mathbf{k}$ is the unit vector in the direction of observation.

Spatial structure of the magnetic field of axisymmetric TM solutions in the presence of a spherical particle with satisfying the Sommerfeld condition at infinity has the form [25]:

$$\begin{aligned} H_\varphi &= h_n^{(1)}(k_0R)j_n\left(k_0 r\sqrt{\varepsilon\mu}\right)P_n^1(\cos\theta), r<R,\\ H_\varphi &= j_n\left(k_0\sqrt{\varepsilon\mu}R\right)h_n^{(1)}(k_0 r)P_n^1(\cos\theta), r>R, \end{aligned} \tag{3}$$

where R is the sphere radius, and it is assumed that the surrounding space is vacuum. In Eq. (3) and further, $j_n(x), h_n^{(1)}(x)$ and $P_n^1(x)$ are the spherical Bessel functions and the Legendre polynomials, correspondingly.

The continuity condition for the tangential components of the solution [Eq. (3)], that is, the dispersion equation for the frequency k_0, in this case has the form [25]:

$$\begin{aligned} &\varepsilon\frac{d}{dz_2}[z_2h_n^{(1)}(z_2)]j_n(z_1) - \frac{d}{dz_1}[z_1j_n(z_1)]h_n^{(1)}(z_2) = 0,\\ &z_1 = \sqrt{\varepsilon\mu}k_0R, z_2 = k_0R. \end{aligned} \tag{4}$$

It is very important that the dispersion Eq. (4) is valid for any values of permittivity and permeability, including DNG particles with $\varepsilon < 0$ and $\mu < 0$.

Figure 1 shows solutions of the dispersion Eq. (4) for electromagnetic eigenmodes in DNG particles in the ideal lossless case [$\mu = -1$, $\varepsilon \in (-60, -1.25)$] and in the more realistic lossy DNG metamaterial case [$\mu = -1 + 0.1i$, $\varepsilon \in (-60, -1.5) + 0.1i$].

It can be seen in Fig. 1 that in the case of DNG particles, there are roots both with negative imaginary parts of frequencies and with positive ones. Such behavior is radically different from the case of a dielectric sphere (see Supplement 1).

Negative values of the frequency imaginary part correspond to exterior oscillations of the electromagnetic field outside the DNG sphere, and they rapidly grow at infinity. Completely analogous quasi-normal mode (QNM) exterior oscillations also take place in the case of a dielectric sphere. As the refractive index tends to infinity, the imaginary part of these modes tends to a finite negative value, determined by the root of the equation $d(zh_n^{(1)}(z))/dz = 0$. At $n = 1$ (dipole mode) this root is

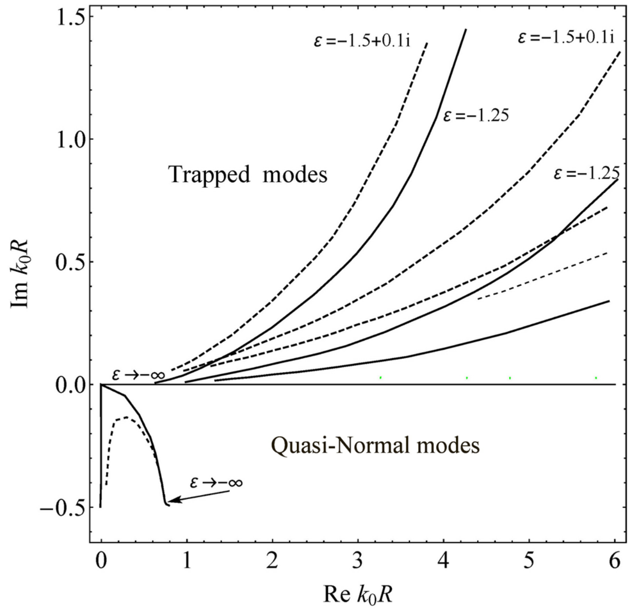

Fig. 1. Dependence of the complex roots of Eq. (4) on the permittivity ε for the ideal lossless case [$\mu = -1$, $\varepsilon \in (-60, -1.25)$, solid line] and for the more realistic case of lossy DNG metamaterial [$\mu = -1 + 0.1i$, $\varepsilon \in (-60, -1.5) + 0.1i$, dashed line]. The sphere is in vacuum. Dipole case, n = 1. Different branches correspond to different radial quantum numbers of the solutions, that is, the number of zeros of the mode field in the radial direction.

$k_0R = 0.866–0.5i$, and the corresponding oscillations are low-Q quasi-normal modes and, therefore, such modes are of little interest.

However, in the case of a DNG sphere, unlike a dielectric sphere, there are natural oscillations with positive imaginary parts of frequencies. Such oscillations, in accordance with the Sommerfeld radiation condition, decay exponentially at infinity. If the refractive index tends to infinity, the imaginary part of the complex frequency of these modes tends to zero, i.e., they are high-quality oscillations which may, therefore, have practical importance. It is very important that the exponential decay takes place not only in the case of lossless DNG metamaterial but also in the more realistic case of DNG metamaterial with losses, which necessarily exist in accordance with the Kramers–Kronig relations [26,27]. In the presence of losses, the exponential decay becomes even stronger (see Fig. 1), and this is quite understandable, since the part of the energy associated with Joule losses in the DNG metamaterial is spent to its heating and not to radiation. Since these modes are strongly localized, they are square integrable, and therefore it is natural to call them trapped modes.

In the region of high refractive indices, $N = \sqrt{\varepsilon\mu} >> 1$, asymptotic solutions of dispersion Eq. (4) for high-Q trapped modes have the form (see Supplement 1 for details):

$$\begin{aligned} &\mathrm{Re}(k_{0,n}R) = \frac{Z_n}{N}\left(1 - \frac{\mu}{nN^2} + \ldots\right), \mathrm{Im}(k_{0,n}R) = -\mu\frac{Z_n^{2n+2}}{N^{2n+4}}\xi_n,\\ &\xi_1 = 1, \xi_2 = 1/36, \xi_3 = 1/2025, \ldots, \end{aligned} \tag{5}$$

where Z_n is the root of the equation $J_{n+1/2}(Z_n) = 0$. It clearly follows from Eq. (5) that when the sign of μ changes, the sign of the imaginary part of the frequency changes also, which leads to exponential damping of the trapped modes.

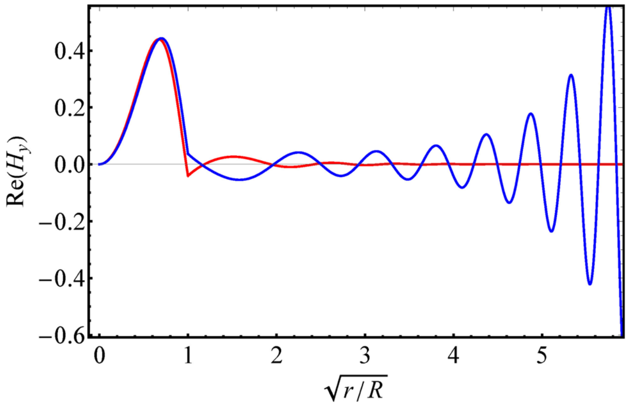

Fig. 2. The dependence of Re($H_y(r, \theta = \pi/2)$) on the radius r for the quasi-normal TM_{101} mode in a dielectric sphere ($\varepsilon = 10$, $\mu = 1$, $k_0R = 1.35715 - 0.160978i$, blue curve) and for the trapped TM_{101} mode in a DNG sphere ($\varepsilon = -10$, $\mu = -1$, $k_0R = 1.46872 + 0.126651i$, red curve).

Figure 2 shows the spatial distributions of the magnetic field for modes in dielectric particles with an ordinary refractive index and in particles with a negative refractive index. Figure 2 clearly shows the exponential decay of trapped TM_{101} ($\varepsilon = -10$, $\mu = -1$) modes and the exponential growth of quasi-normal modes TM_{101} ($\varepsilon = 10$, $\mu = 1$) at infinity. In this case, the fields inside the sphere for dielectric and DNG spheres with refractive indices of the same modulus practically do not differ.

To clarify the physical reasons for the occurrence of trapped modes, Fig. 3 shows the streamlines of the Poynting vector, $\mathbf{S} = c/8\pi \mathrm{Re}(\mathbf{E} \times \mathbf{H}^*)$, for the trapped modes and the quasi-normal modes in the dielectrics.

It can be seen in Fig. 3(a) that due to the negative refraction at the interface between the DNG particle and the vacuum, the streamlines are closed, distinguishing fundamentally the case of the DNG sphere from the case of an ordinary dielectric sphere, where the streamlines are open [Fig. 3(b)]. Thus, the existence of trapped modes in DNG particles is associated with the main property of media with a negative refractive index, that is, with opposite directions of the phase and group velocities of light. This mechanism of the origin of trapped non-radiating modes is fundamentally different from the Friedrich–Wintgen mechanism [28] and other mechanisms [29] which are usually used for the phenomenological description of BIC and consist in searching for conditions of destructive interference of two radiating modes.

In contrast to the diverging quasi-normal modes in dielectric particles, the trapped modes form a complete orthogonal system (see Supplement 1):

$$\left(\varepsilon \int_{V^-} \mathbf{E}_s \mathbf{E}_{s'} dV + \int_{V^+} \mathbf{E}_s \mathbf{E}_{s'} dV \right) = 0, s \neq s', \quad (6)$$

where V^- and V^+ stand for volumes of particle and exterior space, correspondingly.

The trapped modes found for DNG particles are not abstract solutions of the sourceless Maxwell equations. They manifest themselves clearly in the scattering of Bessel beams by a DNG sphere (for general formulas, see Ref. [30]). The symmetry of Bessel beams is in perfect agreement with the symmetry of any axisymmetric bodies and therefore is well suited for studying their non-trivial optical properties. In the case of axisymmetric TM polarization, the only non-zero magnetic field component

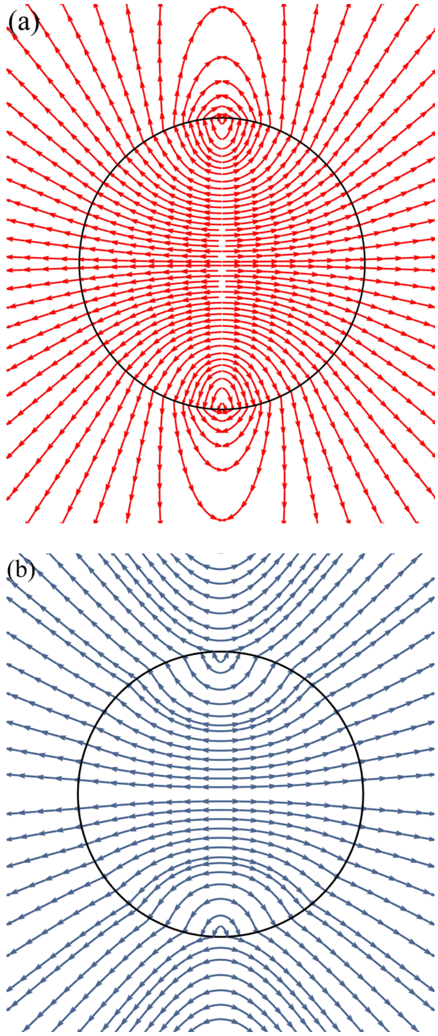

Fig. 3. Streamlines of the Poynting vector (a) for trapped TM_{101} mode in DNG sphere ($k_0R = 1.46872 + 0.126651i$, $\varepsilon = -10$, $\mu = -1$) and (b) for quasi-normal TM_{101} mode in dielectric sphere ($k_0R = 1.35715 - 0.160978i$, $\varepsilon = 10$, $\mu = 1$).

of the Bessel beam has the form:

$$H_\varphi = H_0 J_1(k_0 \rho \sin\beta) e^{ik_0 z \cos\beta} (\text{TM case}). \quad (7)$$

The wave vector components of the Bessel beam [Eq. (7)] form a cone having a conical angle β relative to the z axis. Here, ρ and z stand for cylindrical coordinates.

Figure 4 shows the dependence of the scattered power on the size parameter of the sphere, k_0R, in the case of axisymmetric Bessel beam with TM polarization.

This spectrum shows clearly the presence of the perfect non-radiating modes (PTM) [9–11] and the trapped modes, manifesting themselves as the minima and the maxima of the scattered power, respectively. The shape of the scattering spectrum is generally like the spectrum of scattering by a dielectric sphere, but the modes causing large scattering in this case arise at the frequencies of the trapped modes.

In conclusion, we have shown that in the DNG particles with $\mathrm{Re}\varepsilon < 0$ and $\mathrm{Re}\mu < 0$, there are modes that decay exponentially at infinity. Therefore, these modes can be regarded as trapped modes with a finite energy. The existence of such modes is associated with the main property of the media with a negative

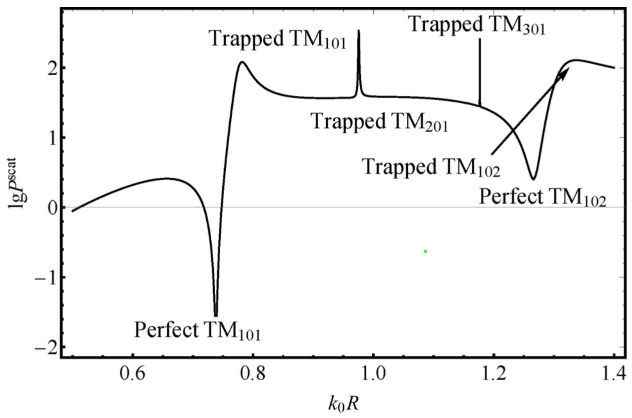

Fig. 4. Dependence of the scattered power on the size parameter of the sphere k_0R (axisymmetric Bessel beam, TM polarization, $\varepsilon = -36$, $\mu = -1$ $\beta = \pi/4$).

refractive index, that is, with opposite directions of the phase and group velocities of light. This mechanism of origin of trapped modes is fundamentally different from the Friedrich–Wintgen mechanism [28], which is usually used for the phenomenological description of BIC and consists in searching for conditions of destructive interference of two modes.

We have proved the existence of trapped modes by the example of a spherical DNG particle and TM polarizations, generalization to non-spherical particles, and other field polarizations can be carried out in complete analogy with the analysis given in this Letter.

Within Schrödinger quantum mechanics, a similar solution is hardly possible. Indeed, in the case of quantum mechanics, the scalar wave function and its normal derivative are continuous at the boundary of the potential well, while in optics the tangential components of the electric and magnetic fields should be continuous. In the case of the spinor Pauli or Dirac equation, it is possible to realize negative refraction [31,32], and therefore trapped modes are possible there.

In this work, the properties of trapped modes in the DNG particles in the monochromatic case were considered. The manifestation of the trapped modes in the non-monochromatic case, taking into account both temporal and spatial dispersion, can be very interesting, since in this case non-trivial aspects of the causality principle and Kramers–Kronig relations [33] can appear.

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Data availability. No data were generated or analyzed in the presented research.

Supplemental document. See Supplement 1 for supporting content.

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